



Symmetric Matrices and Quadratic Forms

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01

Symmetric Matrix

Symmetric Matrix

- A symmetric matrix is a matrix A such that $A^T = A$. Such a matrix is necessarily square. Its main diagonal entries are arbitrary, but its other entries occur in pairs – on opposite sides of the main diagonal.

Symmetric: $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & 8 \\ 0 & 8 & -7 \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

Nonsymmetric: $\begin{bmatrix} 1 & -3 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -4 & 0 \\ -6 & 1 & -4 \\ 0 & -6 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$

Orthogonally Diagonalizable

Definition

A square matrix A is **orthogonally diagonalizable** if its eigenvectors are orthogonal.



Eigenvalues of Symmetric Matrix

Theorem

All the **eigenvalues** of matrix A (a real symmetric matrix) are real.



Proof?

Algebraic & Geometric Multiply

Theorem

For real symmetric matrix:

Geometric multiplicity of λ = Algebraic multiplicity of λ

Proof?



Orthogonally Diagonalizable

Theorem

An $n \times n$ matrix A is **orthogonally diagonalizable** if and only if A is a symmetric matrix.

(\Rightarrow):

$$A = A^T \Rightarrow A = Q\Lambda Q^T, \Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$$

(\Leftarrow):

$$A = A^T \Leftarrow A = Q\Lambda Q^T, \Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}, Q \text{ is orthogonal} \Rightarrow Q^T = Q^{-1}$$

$$A^T = (Q\Lambda Q^{-1})^T = (Q\Lambda Q^T)^T = Q\Lambda^T Q^T = Q\Lambda Q^T = A$$

Relationship between eigenvalue and pivot signs

Theorem

For a symmetric matrix the signs of the pivots are the signs of the eigenvalues.

number of positive pivots = number of positive eigenvalues

- We know that determinant of matrix is product of pivots.
- We know that determinant of matrix is product of eigenvalues.

No Proof!

02

Quadratic Form




Quadratic Form

- A quadratic form is any **homogeneous polynomial of degree two** in any number of variables. In this situation, **homogeneous** means that all the terms are of degree two.
 - For example, the expression $7x_1x_2 + 3x_2x_4$ is homogeneous, but the expression $x_1 - 3x_1x_2$ is not.
 - The square of the distance between two points in an inner-product space is a quadratic form.

Quadratic Form

- Given a square **symmetric** matrix $A \in \mathbb{R}^{n \times n}$ and a vector $x \in \mathbb{R}^n$, the scalar value $x^T A x$ is called a **quadratic form**.


$$x^T A x = \sum_{i=1}^n x_i (A x)_i = \sum_{i=1}^n x_i \left(\sum_{j=1}^n A_{ij} x_j \right) = \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j$$

- A quadratic form on \mathbb{R}^n is a function Q defined on \mathbb{R}^n whose value at a vector x in \mathbb{R}^n can be computed by an expression of the form $Q(x) = x^T A x$, where A is an $n \times n$ symmetric matrix. The matrix A is called the **matrix of the quadratic form**.



Quadratic Form

Definition

- Suppose \mathcal{X} is a vector space over \mathbb{R} . Then a function $Q: \mathcal{X} \rightarrow \mathbb{R}$ is called a quadratic form if there exists a bilinear form $f: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ such that:

$$Q(x) = f(x, x) \text{ for all } x \in \mathcal{X}$$

Example

Simplest example of a nonzero quadratic form is ...

Quadratic Form

Example

Without cross-product term: $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

With cross-product term: $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

Tip

- Quadratic forms are easier to use when they have no cross-product terms; that is, when the matrix of the quadratic form (A) is a diagonal matrix.

Quadratic Form

Example

For x in \mathbb{R}^3 , let $Q(x) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$. Write this quadratic form as $x^T Ax$.

Change of Variable in QF

□ If x represents a variable in \mathbb{R}^n , then a **change of variable** is an equation of the form:

$$x = Py$$

or equivalently,

$$y = P^{-1}x$$



where P is an **invertible matrix** and y is a new variable vector in \mathbb{R}^n .

Note

y can be regarded as the **coordinate vector** of x relative to the basis of \mathbb{R}^n determined by the columns of P .



Change of Variable in QF

- If the change of variable is made in a quadratic form $\mathbf{x}^T A \mathbf{x}$, then


$$\mathbf{x}^T A \mathbf{x} = (P\mathbf{y})^T A (P\mathbf{y}) = \mathbf{y}^T P^T A P \mathbf{y} = \mathbf{y}^T (\textcolor{red}{P}^T A \textcolor{red}{P}) \mathbf{y}$$

- The new matrix of the quadratic form is $P^T A P$.
- A is symmetric, so there is an **orthogonal matrix** P such that $P^T A P$ is a diagonal matrix D .
- Then the quadratic form $\mathbf{x}^T A \mathbf{x}$ becomes $\mathbf{y}^T D \mathbf{y}$. There is **no cross-product**.

Quadratic Form

- If A and B are $n \times n$ real matrices connected by the relation

$$B = \frac{1}{2} (A + A^T)$$



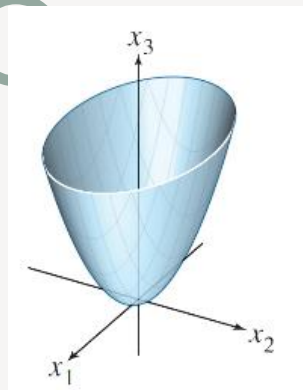
then the corresponding quadratic forms of A and B are identical, and B is symmetric



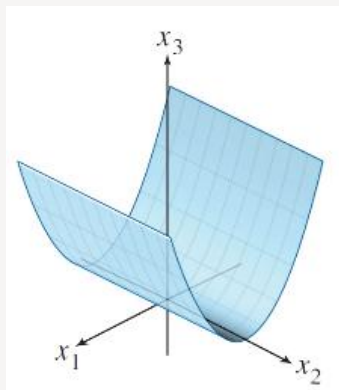
Classifying Quadratic Forms

- When A is an $n \times n$ matrix, the quadratic form $Q(x) = x^T A x$ is a real-valued function with domain \mathbb{R}^n .

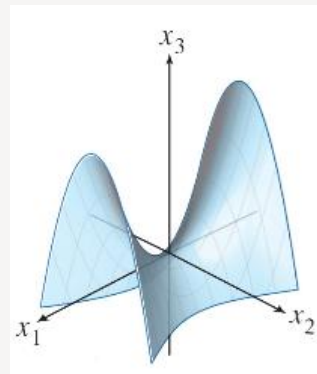
point (x_1, x_2, z) where $z = Q(x)$



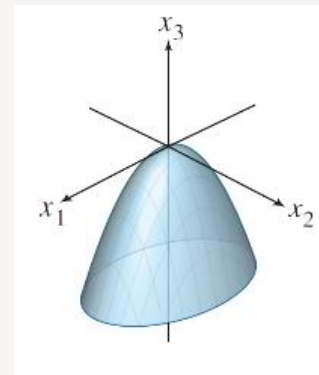
(a) $z = 3x_1^2 + 7x_2^2$



(b) $z = 3x_1^2$



(c) $z = 3x_1^2 - 7x_2^2$



(d) $z = -3x_1^2 - 7x_2^2$ ○

Classifying Quadratic Forms

- A symmetric matrix $A \in \mathbb{S}^n$ is **positive definite (PD)** if for all non zero vectors $A \in \mathbb{R}^n$, $x^T A x > 0$. This is usually denoted $A > 0$, and often times the set of all positive definite matrices is denoted \mathbb{S}_{++}^n .
- A symmetric matrix $A \in \mathbb{S}^n$ is **positive semidefinite (PSD)** if for all vectors $x^T A x \geq 0$. This is written $A \geq 0$, and the set of all positive semidefinite matrices is often denoted \mathbb{S}_+^n .
- Likewise, a symmetric matrix $A \in \mathbb{S}^n$ is **negative definite (ND)**, denoted $A < 0$ if for all non-zero $x \in \mathbb{R}^n$, $x^T A x < 0$.
- Similarly, a symmetric matrix $A \in \mathbb{S}^n$ is **negative semidefinite (NSD)**, denoted $A \leq 0$ if for all $x \in \mathbb{R}^n$, $x^T A x \leq 0$.
- Finally, a symmetric matrix $A \in \mathbb{S}^n$ is **indefinite**, if it is neither positive semidefinite nor negative semidefinite; i.e., if there exists $x_1, x_2 \in \mathbb{R}^n$ such that $x_1^T A x_1 > 0$ and $x_2^T A x_2 < 0$.

Classifying Quadratic Forms

Definition

$$Q(x) = x^T A x$$

A quadratic form Q is:

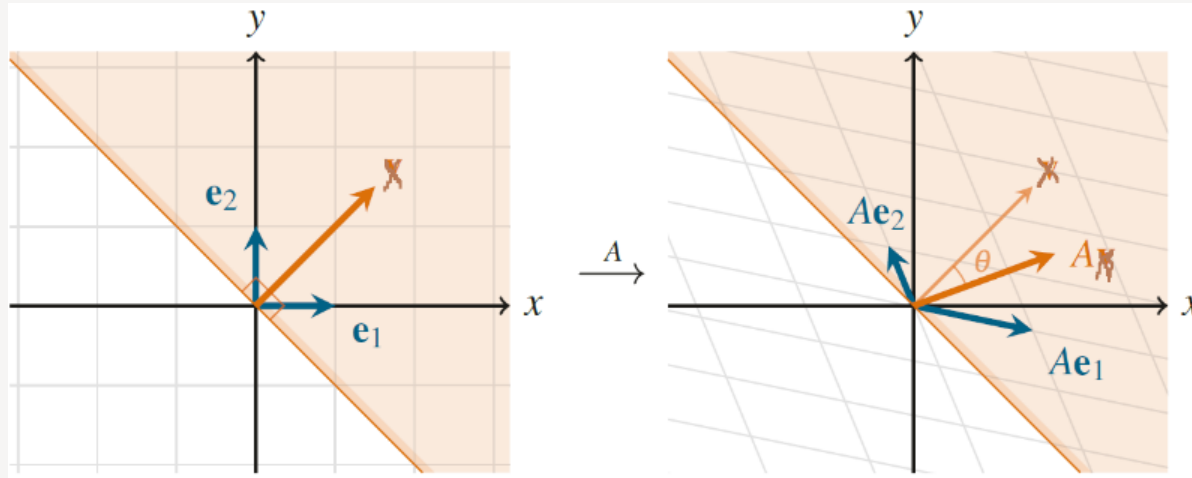
- **positive definite** if $Q(x) > 0$ for all $x \neq 0$;
- **negative definite** if $Q(x) < 0$ for all $x \neq 0$;
- **indefinite** if $Q(x)$ assumes both positive and negative values;
- **positive semidefinite** if $Q(x) \geq 0$ for all x ;
- **negative semidefinite** if $Q(x) \leq 0$ for all x ;

□ For diagonal matrix $A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & & 0 \\ & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \Rightarrow x^T A x = a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2.$

Geometric Interpretation

□ $Q(x) = x^T A x$

□ $\theta = \arccos\left(\frac{(Ax) \cdot x}{\|x\| \|Ax\|}\right)$



Characterization of Positive Definite Matrices

Suppose $A \in \mathcal{M}_n(\mathbb{F})$ is **self-adjoint** ($A^* = A$).. The following are equivalent:

- a) A is positive *definite*.
- b) All of the eigenvalues of A are *strictly positive*.
- c) There is an *invertible* matrix $B \in \mathcal{M}_n(\mathbb{F})$ such that $A = B^*B$
- d) There is a diagonal matrix $D \in \mathcal{M}_n(\mathbb{R})$ with *strictly positive* diagonal entries and a unitary matrix $U \in \mathcal{M}_n(\mathbb{F})$ such that $A = UDU^*$.

You can extend these facts to other categories!

Characterization of Positive Definite Matrices

Suppose $A \in \mathcal{M}_n(\mathbb{F})$ is self-adjoint ($A^* = A$). The following are equivalent:

- a) A is positive semidefinite.
- b) All of the eigenvalues of A are non-negative.
- c) There is a matrix $B \in \mathcal{M}_n(\mathbb{F})$ such that $A = B^*B$, and
- d) There is a diagonal matrix $D \in \mathcal{M}_n(\mathbb{R})$ with non-negative diagonal entries and a unitary matrix $U \in \mathcal{M}_n(\mathbb{F})$ such that $A = UDU^*$.

Quadratic Form

Theorem

Let A be an $n \times n$ symmetric matrix. Then a quadratic form $x^T Ax$ is:

- **positive definite** if and only if the eigenvalues of A are **all positive**;
- **negative definite** if and only if the eigenvalues of A are **all negative**;
- **indefinite** if and only if A has **both positive and negative** eigenvalues;

□ How about semidefinite?


03

Positive Definite Tests



Change of Variable in QF

Five tests to see whether a matrix is positive definite or not:

- 
1. $x^T A x > 0$ for all x (other than zero-vector)
 2. If A is positive definite, $A = S^T S$ (S must have independent columns.)
 3. All eigen values are greater than 0
 4. Sylvester's Criterion: All upper left determinants must be > 0 .
 5. Every pivot must be > 0

Note

A positive definite matrix A has positive eigenvalues, positive pivots, positive determinants, and positive energy $v^T A v$ for every vector v . $A = S^T S$ is always positive definite if S has independent columns.

Positive Definite Matrices

For positive definite matrices we had:

- *If S is positive definite, $S = A^T A$ (A must have independent columns.)*



Theorem

If S is positive definite $S = A^T A$ (A must have independent columns):
 $A^T A$ is positive definite iff the columns of A are linearly independent.

□ Proof?

Positive Definite Matrices

For positive definite matrices we had:

- *All eigen values are greater than 0.*



Theorem

If a matrix is positive definite, then its eigenvalues are positive.

□ Proof?

Theorem

If a matrix has positive eigenvalues, then it is positive definite.


□ Proof?



Positive Definite Matrices

For positive definite matrices we had:

- *Sylvester's Criterion: All upper left determinants must be > 0 .*


$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

The matrix A is shown with three nested boxes highlighting its leading principal minors: a blue box around the top-left element (2), a green box around the top-left 2x2 submatrix, and a red box around the entire 3x3 matrix.

Theorem

If a matrix is positive definite, then it has positive determinant.

□ Proof?



Sylvester's Criterion

Theorem

Then A is positive definite if and only if, for all $1 \leq k \leq n$, the determinant of the top-left $k \times k$ block of A is strictly positive.

□ Proof?

Sylvester's Criterion for Positive Semidefinite Matrices

- A **principal minor** of a square matrix is the determinant of a submatrix of A that is obtained by deleting some (or none) of its rows as well as the corresponding columns.
- A matrix is positive semidefinite if and only if all of its principal minors are non-negative.

$$B = \begin{bmatrix} a & b & c \\ \bar{b} & d & e \\ \bar{c} & \bar{e} & f \end{bmatrix}$$

are $a, d, f, \det(B)$ itself, as well as

$$\det \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} = ad - |b|^2$$

$$\det \begin{pmatrix} a & c \\ \bar{c} & f \end{pmatrix} = af - |c|^2$$

$$\det \begin{pmatrix} d & e \\ \bar{e} & f \end{pmatrix} = df - |e|^2$$

Pivots & Positive Definite Matrices

Theorem

If a matrix has positive pivots, then it is positive definite.



□ Proof?

Properties

Important

- If A is positive definite, A^{-1} will also be positive definite.
- If A and B are positive definite matrices, $A + B$ will also be a positive definite matrix.
- Positive definite and negative definite matrices are always full rank, and hence, invertible.
- For $A \in \mathbb{R}^{m \times n}$ gram matrix is always positive semidefinite. Further, if $m \geq n$ (and we assume for convenience that A is full rank), then gram matrix is positive definite.

Properties

Important

Suppose $A, B \in \mathcal{M}_n$ are positive (semi)definite, $P \in \mathcal{M}_{n,m}$ is any matrix, and $c > 0$ is real scalar. Then

- a) $A + B$ is positive (semi)definite.
- b) cA is positive (semi)definite.
- c) A^T is positive (semi)definite, and
- d) P^*AP is positive semidefinite. Furthermore, if A is positive definite then P^*AP is positive definite if and only if $\text{rank}(P) = m$.

04

Gram Matrix



Gram Matrix

$$\text{Gram}(A) : A^T A$$

- symmetric
- non-negative eigenvalues
- real eigenvalues
- orthonormal eigenvectors
- positive semi-definite

Proof?

