



# Symmetric Matrices and Quadratic Forms

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# Symmetric Matrix

# Symmetric Matrix

- A symmetric matrix is a matrix  $A$  such that  $A^T = A$ . Such a matrix is necessarily square. Its main diagonal entries are arbitrary, but its other entries occur in pairs – on opposite sides of the main diagonal.

Symmetric:  $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & 8 \\ 0 & 8 & -7 \end{bmatrix}$ ,  $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

Nonsymmetric:  $\begin{bmatrix} 1 & -3 \\ 3 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & -4 & 0 \\ -6 & 1 & -4 \\ 0 & -6 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} 5 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$

# Orthogonally Diagonalizable

## Definition

A square matrix  $A$  is **orthogonally diagonalizable** if its eigenvectors are orthogonal.



# Eigenvalues of Symmetric Matrix

## Theorem

All the **eigenvalues** of matrix A (a real symmetric matrix) are **real**.

Proof?

# Algebraic & Geometric Multiply

## Theorem

For real symmetric matrix:

**Geometric multiplicity of  $\lambda$  = Algebraic multiplicity of  $\lambda$**

Proof?

# Orthogonally Diagonalizable

## Theorem

An  $n \times n$  matrix  $A$  is **orthogonally diagonalizable** if and only if  $A$  is a symmetric matrix.

( $\Rightarrow$ ):

$$A = A^T \Rightarrow A = Q\Lambda Q^T, \Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$$

( $\Leftarrow$ ):

$$A = A^T \Leftarrow A = Q\Lambda Q^T, \Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}, Q \text{ is orthogonal} \Rightarrow Q^T = Q^{-1}$$

$$A^T = (Q\Lambda Q^{-1})^T = (Q\Lambda Q^T)^T = Q\Lambda^T Q^T = Q\Lambda Q^T = A$$

# Relationship between eigenvalue and pivot signs

## Theorem

For a symmetric matrix the signs of the pivots are the signs of the eigenvalues.

*number of positive pivots=number of positive eigenvalues*

- We know that determinant of matrix is product of pivots.
- We know that determinant of matrix is product of eigenvalues.

No Proof!

02

# Quadratic Form



# Quadratic Form

- A quadratic form is any **homogeneous polynomial of degree two** in any number of variables. In this situation, **homogeneous** means that all the terms are of degree two.
  - For example, the expression  $7x_1x_2 + 3x_2x_4$  is homogeneous, but the expression  $x_1 - 3x_1x_2$  is not.
  - The square of the distance between two points in an inner-product space is a quadratic form.

# Quadratic Form

- Given a square **symmetric** matrix  $A \in \mathbb{R}^{n \times n}$  and a vector  $x \in \mathbb{R}^n$ , the scalar value  $x^T A x$  is called a **quadratic form**.

$$x^T A x = \sum_{i=1}^n x_i (Ax)_i = \sum_{i=1}^n x_i \left( \sum_{j=1}^n A_{ij} x_j \right) = \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j$$

- A quadratic form on  $\mathbb{R}^n$  is a function  $Q$  defined on  $\mathbb{R}^n$  whose value at a vector  $x$  in  $\mathbb{R}^n$  can be computed by an expression of the form  $Q(x) = x^T A x$ , where  $A$  is an  $n \times n$  symmetric matrix. The matrix  $A$  is called the **matrix of the quadratic form**.

# Quadratic Form

## Definition

- Suppose  $\mathcal{X}$  is a vector space over  $\mathbb{R}$ . Then a function  $Q: \mathcal{X} \rightarrow \mathbb{R}$  is called a quadratic form if there exists a bilinear form  $f: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  such that:

$$Q(x) = f(x, x) \text{ for all } x \in \mathcal{X}$$

## Example

Simplest example of a nonzero quadratic form is ...

# Quadratic Form

## Example

Without cross-product term:

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

With cross-product term:

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$$

## Tip

- Quadratic forms are easier to use when they have no cross-product terms; that is, when the matrix of the quadratic form (A) is a diagonal matrix.

# Quadratic Form

## Example

For  $x$  in  $\mathbb{R}^3$ , let  $Q(x) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$ . Write this quadratic form as  $x^T Ax$ .

# Change of Variable in QF

- If  $x$  represents a variable in  $\mathbb{R}^n$ , then a **change of variable** is an equation of the form:

$$x = Py$$

or equivalently,

$$y = P^{-1}x$$

where  $P$  is an **invertible matrix** and  $y$  is a new variable vector in  $\mathbb{R}^n$ .

## Note

$y$  can be regarded as the **coordinate vector** of  $x$  relative to the basis of  $\mathbb{R}^n$  determined by the columns of  $P$ .

# Change of Variable in QF

- If the change of variable is made in a quadratic form  $\mathbf{x}^T A \mathbf{x}$ , then

$$\mathbf{x}^T A \mathbf{x} = (P\mathbf{y})^T A (P\mathbf{y}) = \mathbf{y}^T P^T A P \mathbf{y} = \mathbf{y}^T (\mathbf{P}^T \mathbf{A} \mathbf{P}) \mathbf{y}$$

- The new matrix of the quadratic form is  $P^T A P$ .
- $A$  is symmetric, so there is an **orthogonal matrix**  $P$  such that  $P^T A P$  is a diagonal matrix  $D$ .
- Then the quadratic form  $\mathbf{x}^T A \mathbf{x}$  becomes  $\mathbf{y}^T D \mathbf{y}$ . There is **no cross-product**.

# Quadratic Form

- If  $A$  and  $B$  are  $n \times n$  real matrices connected by the relation

$$B = \frac{1}{2} (A + A^T)$$

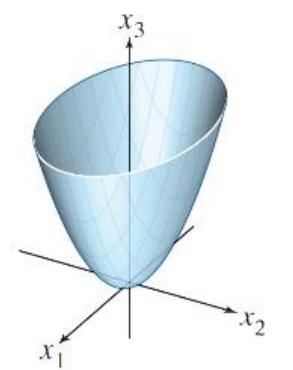
then the corresponding quadratic forms of  $A$  and  $B$  are identical, and  $B$  is symmetric

# Classifying Quadratic Forms

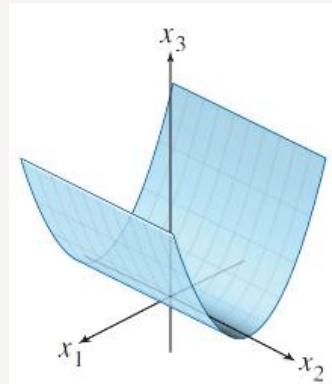
- When  $A$  is an  $n \times n$  matrix, the quadratic form  $Q(x) = x^T A x$  is a real-valued function with domain  $\mathbb{R}^n$ .

point  $(x_1, x_2, z)$  where  $z = Q(x)$

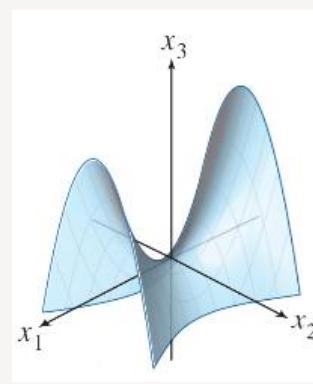
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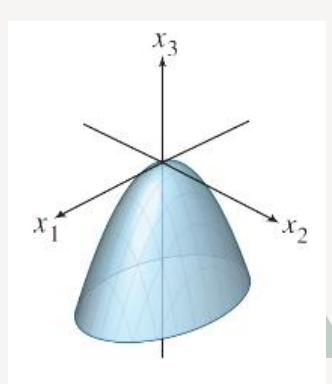
(a)  $z = 3x_1^2 + 7x_2^2$



(b)  $z = 3x_1^2$



(c)  $z = 3x_1^2 - 7x_2^2$



(d)  $z = -3x_1^2 - 7x_2^2$

○

# Classifying Quadratic Forms

- A symmetric matrix  $A \in \mathbb{S}^n$  is **positive definite (PD)** if for all non-zero vectors  $x \in \mathbb{R}^n$ ,  $x^T A x > 0$ . This is usually denoted  $A > 0$ , and often times the set of all positive definite matrices is denoted  $\mathbb{S}_{++}^n$ .
- A symmetric matrix  $A \in \mathbb{S}^n$  is **positive semidefinite (PSD)** if for all vectors  $x^T A x \geq 0$ . This is written  $A \geq 0$ , and the set of all positive semidefinite matrices is often denoted  $\mathbb{S}_+^n$ .
- Likewise, a symmetric matrix  $A \in \mathbb{S}^n$  is **negative definite (ND)**, denoted  $A < 0$  if for all non-zero  $x \in \mathbb{R}^n$ ,  $x^T A x < 0$ .
- Similarly, a symmetric matrix  $A \in \mathbb{S}^n$  is **negative semidefinite (NSD)**, denoted  $A \leq 0$  if for all  $x \in \mathbb{R}^n$ ,  $x^T A x \leq 0$ .
- Finally, a symmetric matrix  $A \in \mathbb{S}^n$  is **indefinite**, if it is neither positive semidefinite nor negative semidefinite; i.e., if there exists  $x_1, x_2 \in \mathbb{R}^n$  such that  $x_1^T A x_1 > 0$  and  $x_2^T A x_2 < 0$ .

# Classifying Quadratic Forms

## Definition

$$Q(x) = x^T A x$$

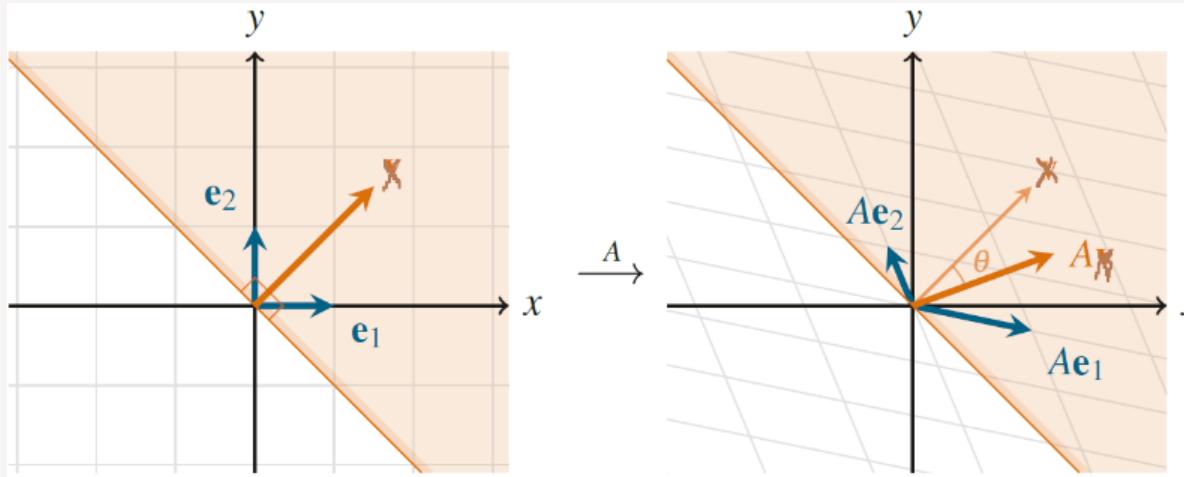
A quadratic form  $Q$  is:

- **positive definite** if  $Q(x) > 0$  for all  $x \neq 0$ ;
- **negative definite** if  $Q(x) < 0$  for all  $x \neq 0$ ;
- **indefinite** if  $Q(x)$  assumes both positive and negative values;
- **positive semidefinite** if  $Q(x) \geq 0$  for all  $x$ ;
- **negative semidefinite** if  $Q(x) \leq 0$  for all  $x$ ;

□ For diagonal matrix  $A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \Rightarrow x^T A x = a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2.$

# Geometric Interpretation

- $Q(x) = x^T A x$
- $\theta = \arccos\left(\frac{(Ax) \cdot x}{\|x\| \|Ax\|}\right)$



# Characterization of Positive Definite Matrices

Suppose  $A \in \mathcal{M}_n(\mathbb{F})$  is **self-adjoint** ( $A^* = A$ ). The following are equivalent:

- a)  $A$  is positive *definite*.
- b) All of the eigenvalues of  $A$  are *strictly positive*.
- c) There is an *invertible* matrix  $B \in \mathcal{M}_n(\mathbb{F})$  such that  $A = B^*B$
- d) There is a diagonal matrix  $D \in \mathcal{M}_n(\mathbb{R})$  with *strictly positive* diagonal entries and a unitary matrix  $U \in \mathcal{M}_n(\mathbb{F})$  such that  $A = UDU^*$ .

You can extend these facts to other categories!

# Characterization of Positive Definite Matrices

Suppose  $A \in \mathcal{M}_n(\mathbb{F})$  is **self-adjoint** ( $A^* = A$ ). The following are equivalent:

- a)  $A$  is positive semidefinite.
- b) All of the eigenvalues of  $A$  are non-negative.
- c) There is a matrix  $B \in \mathcal{M}_n(\mathbb{F})$  such that  $A = B^*B$ , and
- d) There is a diagonal matrix  $D \in \mathcal{M}_n(\mathbb{R})$  with non-negative diagonal entries and a unitary matrix  $U \in \mathcal{M}_n(\mathbb{F})$  such that  $A = UDU^*$ .

# Quadratic Form

## Theorem

Let  $A$  be an  $n \times n$  symmetric matrix. Then a quadratic form  $x^T A x$  is:

- **positive definite** if and only if the eigenvalues of  $A$  are **all positive**;
- **negative definite** if and only if the eigenvalues of  $A$  are **all negative**;
- **indefinite** if and only if  $A$  has **both positive and negative eigenvalues**;

□ How about semidefinite?

03

# Positive Definite Tests



# Change of Variable in QF

Five tests to see whether a matrix is positive definite or not:

1.  $x^T A x > 0$  for all  $x$  (other than zero-vector)
2. If  $A$  is positive definite,  $A = S^T S$  ( $S$  must have independent columns.)
3. All eigen values are greater than 0
4. Sylvester's Criterion: All upper left determinants must be  $> 0$ .
5. Every pivot must be  $> 0$

## Note

A positive definite matrix  $A$  has positive eigenvalues, positive pivots, positive determinants, and positive energy  $v^T A v$  for every vector  $v$ .  $A = S^T S$  is always positive definite if  $S$  has independent columns.

# Positive Definite Matrices

For positive definite matrices we had:

- *If  $S$  is positive definite,  $S = A^T A$  ( $A$  must have independent columns.)*

## Theorem

If  $S$  is positive definite  $S = A^T A$  ( $A$  must have independent columns):  
 $A^T A$  is positive definite iff the columns of  $A$  are linearly independent.

□ Proof?

# Positive Definite Matrices

For positive definite matrices we had:

- *All eigen values are greater than 0.*

## Theorem

If a matrix is positive definite, then its eigenvalues are positive.

□ Proof?

## Theorem

If a matrix has positive eigenvalues, then it is positive definite.

□ Proof?

# Positive Definite Matrices

For positive definite matrices we had:

- *Sylvester's Criterion: All upper left determinants must be  $> 0$ .*

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

## Theorem

If a matrix is positive definite, then it has positive determinant.

□ Proof?

# Sylvester's Criterion

## Theorem

Then  $A$  is positive definite if and only if, for all  $1 \leq k \leq n$ , the determinant of the top-left  $k \times k$  block of  $A$  is strictly positive.

□ Proof?

# Sylvester's Criterion for Positive Semidefinite Matrices

- A **principal minor** of a square matrix is the determinant of a submatrix of  $A$  that is obtained by deleting some (or none) of its rows as well as the corresponding columns.
- A matrix is positive semidefinite if and only if all of its principal minors are non-negative.

$$B = \begin{bmatrix} a & b & c \\ \bar{b} & d & e \\ \bar{c} & \bar{e} & f \end{bmatrix}$$

are  $a, d, f, \det(B)$  itself, as well as

$$\det \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} = ad - |b|^2$$

$$\det \begin{pmatrix} a & c \\ \bar{c} & f \end{pmatrix} = af - |c|^2$$

$$\det \begin{pmatrix} d & e \\ \bar{e} & f \end{pmatrix} = df - |e|^2$$

# Pivots & Positive Definite Matrices

## Theorem

If a matrix has positive pivots, then it is positive definite.

□ Proof?

# Properties

## Important

- ◻ If  $A$  is positive definite,  $A^{-1}$  will also be positive definite.
- ◻ If  $A$  and  $B$  are positive definite matrices,  $A + B$  will also be a positive definite matrix.
- ◻ Positive definite and negative definite matrices are always full rank, and hence, invertible.
- ◻ For  $A \in \mathbb{R}^{m \times n}$  gram matrix is always positive semidefinite. Further, if  $m \geq n$  (and we assume for convenience that  $A$  is full rank), then gram matrix is positive definite.

# Properties

## Important

Suppose  $A, B \in \mathcal{M}_n$  are positive (semi)definite,  $P \in \mathcal{M}_{n,m}$  is any matrix, and  $c > 0$  is real scalar. Then

- a)  $A + B$  is positive (semi)definite.
- b)  $cA$  is positive (semi)definite.
- c)  $A^T$  is positive (semi)definite, and
- d)  $P^*AP$  is positive semidefinite. Furthermore, if  $A$  is positive definite then  $P^*AP$  is positive definite if and only if  $\text{rank}(P) = m$ .

# 04

# Gram Matrix



# Gram Matrix

Gram(A) :  $A^T A$

- ❑ symmetric
- ❑ non-negative eigenvalues
- ❑ real eigenvalues
- ❑ orthonormal eigenvectors
- ❑ positive semi-definite

Proof?